

BOUNDARY NULL CONTROLLABILITY FOR A HEAT EQUATION WITH GENERAL DYNAMICAL BOUNDARY CONDITIONS

UMBERTO BICCARI AND MAHAMADI WARMA

ABSTRACT. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary Γ . Let $\gamma > 0$, $\delta \geq 0$ be real numbers and β a nonnegative measurable function in $L^\infty(\Gamma)$. Using some suitable Carleman estimates, we show that the linear heat equation $\partial_t u - \gamma \Delta u = 0$ in $\Omega \times (0, T)$ with the non-homogeneous general dynamic boundary conditions $\partial_t u_\Gamma - \delta \Delta_\Gamma u_\Gamma + \gamma \partial_\nu u + \beta u_\Gamma = g$ on $\Gamma \times (0, T)$ is always null controllable from the boundary for every $T > 0$ and initial data $(u_0, u_{\Gamma,0}) \in L^2(\Omega) \times L^2(\Gamma)$.

1. INTRODUCTION AND MAIN RESULTS

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary $\Gamma := \partial\Omega$. In the present paper we consider the following heat equation with dynamical boundary conditions

$$\begin{cases} \partial_t u - \gamma \Delta u = 0 & \text{in } \Omega \times (0, T) := \Omega_T \\ \partial_t u_\Gamma - \delta \Delta_\Gamma u_\Gamma + \gamma \partial_\nu u + \beta(x) u_\Gamma = g & \text{on } \Gamma \times (0, T) := \Sigma_T \\ (u, u_\Gamma)|_{t=0} = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \quad (1.1)$$

Here, $\gamma > 0$ and $\delta \geq 0$ are real numbers, β is a nonnegative measurable function which belongs to $L^\infty(\Gamma)$, $\partial_\nu u$ is the normal derivative of u and u_Γ denotes the trace on Γ of the function u , whereas $u_0 \in L^2(\Omega)$, $u_{\Gamma,0} \in L^2(\Gamma)$ and Δ_Γ denotes the Laplace-Beltrami operator on Γ . We emphasize that $u_{\Gamma,0}$ is not necessarily the trace of u_0 , since we do not assume that u_0 has a trace. But if u_0 has a well-defined trace on Γ , then the trace must coincide with $u_{\Gamma,0}$. For example if the one dimensional case $N = 1$, since solutions of (1.1) are continuous on $\overline{\Omega}$, then in that case we have that $u_{\Gamma,0}$ is the trace of u_0 .

Several authors have studied the existence, uniqueness and the regularity of solutions to the system (1.1). We refer for example to the papers [1, 4, 5, 7, 12, 13, 16, 17] and their references. This type of boundary conditions has been also called generalized Wentzell or generalized Wentzell-Robin boundary conditions.

The main concern in the present paper is the investigate the null controllability of the system (1.1) from the boundary, that is, given $T > 0$ and initial data $(u_0, u_{\Gamma,0}) \in L^2(\Omega) \times L^2(\partial\Omega)$, is there a control function $g \in L^2(\Sigma_T)$ such that the unique mild solution (see Definition 1.2 below) satisfies $u(\cdot, T) = 0$ in Ω and $u_\Gamma(\cdot, T) = 0$ on Γ ?

The interior null controllability of the system (1.1) for the case $\delta > 0$ has been recently studied in [13], that is, the case where $g = 0$ and the first equation is replaced by $\partial_t u - \gamma \Delta u = f|_\omega$ in Ω_T . The authors have shown that if $\delta > 0$, then for every open set $\omega \Subset \Omega$, $T > 0$, and initial data $(u_0, u_{\Gamma,0}) \in L^2(\Omega) \times L^2(\partial\Omega)$, there is a control function $f \in L^2(\Omega_T)$ such that the unique mild

2010 *Mathematics Subject Classification.* 93B05, 35K20, 93B07.

Key words and phrases. Heat equation, general dynamic boundary condition, observability inequality, exact controllability from the boundary.

The work of the authors is partially supported by the Air Force Office of Scientific Research under the Award No: FA9550-15-1-0027.

solution of the associated system satisfies $u(\cdot, T) = 0$ in Ω and $u_\Gamma(\cdot, T) = 0$ on Γ . On the other hand, in the one dimensional case, that is, $\Omega = (0, 1)$, it has been proved in [12] that the system (1.1) is approximately controllable, that is, for every initial data $(u_0, u_{0,0}, u_{1,0}) \in L^2(0, 1) \times \mathbb{C}^2$, $T > 0$ and $\varepsilon > 0$, there exists a control function $g \in L^2(0, T)$ such that the unique mild solution u satisfies

$$\|u(\cdot, T) - u_0\|_{L^2(0,1)} + |u(0, T) - u_{0,0}| + |u(1, T) - u_{1,0}| < \varepsilon.$$

We emphasize that the same approximate controllability of the system (1.1) can be proved in the N -dimensional setting, but this will be a simple consequence of the stronger result obtained in the present paper. Since interior null controllability of a system does not imply the null controllability of the system from the boundary, and as approximate controllability does not imply null controllability (but the converse is always true), we have that the results obtained in the present papers will complete the ones contained in [13] and will trivially imply the ones obtained in [12]. Our main result (see Theorem 1.6 below) states that the system is null controllable from the boundary for every $T > 0$ and initial data $(u_0, u_{\Gamma,0}) \in L^2(\Omega) \times L^2(\partial\Omega)$. Its proof is based on suitable Carleman estimates (Theorem 2.1) for solutions of the adjoint system associated with (1.1) which are also used to establish an observability inequality for solutions of the adjoint system. The obtained observability inequality is as usual equivalent to the null controllability of the system. We also notice that our results also include the case $\delta = 0$, that is, when there is no surface diffusion at the boundary. This case has not been considered in the study of the interior controllability in [13].

1.1. The functional setup. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary Γ . For $r, q \in [1, \infty]$ with $1 \leq r, q < \infty$ or $r = q = \infty$ we endow the Banach space

$$\mathbb{X}^{r,q}(\overline{\Omega}) := L^r(\Omega) \times L^q(\Gamma) = \{(f, g) : f \in L^r(\Omega), g \in L^q(\Gamma)\}$$

with the norm

$$\|(f, g)\|_{\mathbb{X}^{r,q}(\overline{\Omega})} := \begin{cases} \|f\|_{L^r(\Omega)} + \|g\|_{L^q(\Gamma)} & \text{if } 1 \leq r \neq q < \infty \\ (\|f\|_{L^r(\Omega)}^r + \|g\|_{L^q(\Gamma)}^q)^{\frac{1}{r}} & \text{if } 1 \leq r = q < \infty \end{cases}$$

and

$$\|(f, g)\|_{\mathbb{X}^{\infty,\infty}(\overline{\Omega})} := \max\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Gamma)}\}.$$

We will simply write $\mathbb{X}^r(\overline{\Omega}) := \mathbb{X}^{r,r}(\overline{\Omega})$. We notice that $\mathbb{X}^r(\overline{\Omega})$ can be identified with the Lebesgue space $L^r(\overline{\Omega}, \mu)$ where the measure μ on $\overline{\Omega}$ is defined for every measurable set $B \subset \overline{\Omega}$ by

$$\mu(B) := |\Omega \cap B| + \sigma(B \cap \Gamma).$$

Here $|\cdot|$ denotes the N -dimensional Lebesgue measure on Ω and σ is the $(N-1)$ -dimensional Lebesgue surface measure on Γ . In addition we have that $\mathbb{X}^2(\overline{\Omega})$ is a Hilbert space with the scalar product

$$\langle (f_1, g_1), (f_2, g_2) \rangle_{\mathbb{X}^2(\overline{\Omega})} = \langle f_1, f_2 \rangle_{L^2(\Omega)} + \langle g_1, g_2 \rangle_{L^2(\Gamma)} = \int_{\Omega} f_1 f_2 \, dx + \int_{\Gamma} g_1 g_2 \, d\sigma.$$

Let

$$W^{1,2}(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\Omega} |\nabla u|^2 \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{W^{1,2}(\Omega)} := \left(\int_{\Omega} |u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}$$

be the first order Sobolev space. The space $W^{1,2}(\Gamma)$ is defined similarly by

$$W^{1,2}(\Gamma) = \left\{ u \in L^2(\Gamma) : \int_{\Gamma} |\nabla_{\Gamma} u|^2 d\sigma < \infty \right\},$$

where ∇_{Γ} denotes the Riemannian gradient (see Section 1.2 below). We also introduce the fractional order Sobolev space

$$W^{\frac{1}{2},2}(\Gamma) := \left\{ u \in L^2(\Gamma) : \int_{\Gamma} \int_{\Gamma} \frac{|u(x) - u(y)|^2}{|x - y|^N} d\sigma_x d\sigma_y < \infty \right\}$$

and we endow it with the norm

$$\|u\|_{W^{\frac{1}{2},2}(\Gamma)} = \left(\int_{\Gamma} |u|^2 d\sigma + \int_{\Gamma} \int_{\Gamma} \frac{|u(x) - u(y)|^2}{|x - y|^N} d\sigma_x d\sigma_y \right)^{\frac{1}{2}}.$$

Since Ω is assumed to have a Lipschitz continuous boundary, then we have the continuous embedding $W^{1,2}(\Omega) \hookrightarrow W^{\frac{1}{2},2}(\Gamma)$. That is, there exists a constant $C > 0$ such that for every $u \in W^{1,2}(\Omega)$, we have

$$\|u_{\Gamma}\|_{W^{\frac{1}{2},2}(\Gamma)} \leq C \|u\|_{W^{1,2}(\Omega)}.$$

For a real number $\delta \geq 0$ we let

$$\mathbb{W}_{\delta}^{1,2}(\overline{\Omega}) := \{U := (u, u_{\Gamma}) : u \in W^{1,2}(\Omega), \delta u \in W^{1,2}(\Gamma)\},$$

and we endow it with the norm

$$\|(u, u_{\Gamma})\|_{\mathbb{W}_{\delta}^{1,2}(\overline{\Omega})} := \left(\|u\|_{W^{1,2}(\Omega)}^2 + \|u_{\Gamma}\|_{W^{1,2}(\Gamma)}^2 \right)^{\frac{1}{2}} \quad \text{if } \delta > 0,$$

and

$$\|(u, u_{\Gamma})\|_{\mathbb{W}_0^{1,2}(\overline{\Omega})} := \left(\|u\|_{W^{1,2}(\Omega)}^2 + \|u_{\Gamma}\|_{W^{\frac{1}{2},2}(\Gamma)}^2 \right)^{\frac{1}{2}} \quad \text{if } \delta = 0.$$

By definition, for every $\delta \geq 0$, we have the continuous embedding $\mathbb{W}_{\delta}^{1,2}(\overline{\Omega}) \hookrightarrow \mathbb{X}^2(\overline{\Omega})$.

1.2. The Laplace-Beltrami operator. We present some basic notion on the Laplace-Beltrami operator. Recall that the boundary Γ of the open set $\Omega \subset \mathbb{R}^N$ can be viewed as a Riemannian manifold endowed with the natural metric inherited from \mathbb{R}^N , given in local coordinates by $\sqrt{\det G} dy_1 \cdots dy_{N-1}$, where $G = (g_{ij})$ denotes the metric tensor. Let ∇_{Γ} be the Riemannian gradient. Then the so called Laplace-Beltrami operator Δ_{Γ} can be at first defined for $u, v \in C^2(\Gamma)$ by the formula

$$-\int_{\Gamma} v \Delta_{\Gamma} u d\sigma = \int_{\Gamma} \langle \nabla_{\Gamma} u, \nabla_{\Gamma} v \rangle_{\Gamma} d\sigma, \quad (1.2)$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ is the Riemannian inner product of tangential vectors on Γ . Throughout the following we shall just denotes $\langle \nabla_{\Gamma} u, \nabla_{\Gamma} v \rangle_{\Gamma} = \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v$. Letting $(g^{ij}) = (g_{ij})^{-1}$, then Δ_{Γ} is given in local coordinates by

$$\Delta_{\Gamma} u = \frac{1}{\sqrt{\det G}} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial y_i} \left(\sqrt{\det G} g^{ij} \frac{\partial u}{\partial y_j} \right). \quad (1.3)$$

Using (1.3) we have that Δ_{Γ} can be considered as a bounded linear operator from $W^{s+2,2}(\Gamma)$ to $W^{s,2}(\Gamma)$, for any $s \in \mathbb{R}$. This implies that the formula (1.2) extends by density to $u, v \in W^{1,2}(\Gamma)$,

where the integral in the left-hand side is to be interpreted in the distributional sense, that is, as $\Delta_\Gamma u \in W^{-1,2}(\Gamma) := (W^{1,2}(\Gamma))^*$. Letting

$$D(\Delta_\Gamma) := \{u \in W^{1,2}(\Gamma), \Delta_\Gamma u \in L^2(\Gamma)\},$$

we have that $-\Delta_\Gamma$ is a self-adjoint and nonnegative operator on $L^2(\Gamma)$ (see, e.g., [14, p. 309]). This implies that Δ_Γ generates a strongly continuous semigroup on $L^2(\Gamma)$ which is also analytic. If Γ is smooth (say of class C^2), then one can show that $D(\Delta_\Gamma) = W^{2,2}(\Gamma)$.

Finally, for the sake of completeness, we mention that in this paper we will never use the local formula (1.3) for the Laplace-Beltrami operator, but rather the so-called surface divergence theorem given in (1.2). Moreover, we recall the following interpolation inequality (see [15, Theorem 1.3.3]). There exists a constant $C > 0$ such that the estimate

$$\|u\|_{W^{1,2}(\Gamma)}^2 \leq C \|u\|_{L^2(\Gamma)} \|u\|_{D(\Delta_\Gamma)} \quad (1.4)$$

holds for every $u \in D(\Delta_\Gamma)$, where the space $D(\Delta_\Gamma)$ is endowed with the graph norm defined by

$$\|u\|_{D(\Delta_\Gamma)} = \|u\|_{L^2(\Gamma)} + \|\Delta_\Gamma u\|_{L^2(\Gamma)}.$$

For more details on this topic we refer to [11, Chapter 3] or [14, Sections 2.4 and 5.1] and their references.

1.3. The well-posedness. In this (sub)section we discuss the well-posedness of the system (1.1). First, let \mathcal{E}_δ be the bilinear symmetric form on $\mathbb{X}^2(\overline{\Omega})$ with domain $D(\mathcal{E}_\delta) := \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ and given for every $U := (u, u_\Gamma), V := (v, v_\Gamma) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ by

$$\mathcal{E}_\delta(U, V) = \gamma \int_\Omega \nabla u \cdot \nabla v \, dx + \delta \int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma v_\Gamma \, d\sigma + \int_\Gamma \beta(x) u_\Gamma v_\Gamma \, d\sigma. \quad (1.5)$$

We assume that $\beta \in L^\infty(\Gamma)$ is measurable and there exists a constant $\beta_0 > 0$ such that

$$\beta(x) \geq \beta_0 \quad \sigma\text{-a.e. on } \Gamma. \quad (1.6)$$

It is well-known that the form \mathcal{E}_δ is closed in $\mathbb{X}^2(\overline{\Omega})$, continuous and elliptic. Under the assumption (1.6), it is also coercive, that is, there is a constant $C > 0$ such that for every $U = (u, u_\Gamma) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$, we have

$$\|U\|_{\mathbb{X}^2(\overline{\Omega})}^2 = \int_\Omega |u|^2 \, dx + \int_\Gamma |u_\Gamma|^2 \, d\sigma \leq C \mathcal{E}_\delta(U, U). \quad (1.7)$$

Second, let A_δ be the linear self-adjoint operator in $\mathbb{X}^2(\overline{\Omega})$ associated with \mathcal{E}_δ in the sense that

$$\begin{cases} D(A_\delta) = \{U \in \mathbb{W}_\delta^{1,2}(\overline{\Omega}), \exists F \in \mathbb{X}^2(\overline{\Omega}), \mathcal{E}_\delta(U, \Phi) = \langle F, \Phi \rangle_{\mathbb{X}^2(\overline{\Omega})}, \forall \Phi \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})\} \\ A_\delta U = -F. \end{cases} \quad (1.8)$$

The following characterization of the operator A_δ can be found in [17].

$$\begin{cases} D(A_\delta) = \{U := (u, u_\Gamma) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega}), \Delta u \in L^2(\Omega), \delta \Delta_\Gamma u_\Gamma - \partial_\nu u \text{ exists in } L^2(\Gamma)\} \\ A_\delta U = \left(\gamma \Delta u, \delta \Delta_\Gamma u_\Gamma - \gamma \partial_\nu u - \beta u_\Gamma \right), \end{cases} \quad (1.9)$$

that is, on its domain, A_δ is the matrix operator

$$A_\delta = \begin{pmatrix} \gamma \Delta & 0 \\ -\gamma \partial_\nu & \delta \Delta_\Gamma - \beta \end{pmatrix}.$$

Throughout the remainder of the article for a function $F = (f, g) \in \mathbb{X}^2(\overline{\Omega})$, by $F \geq 0$, we mean that $f \geq 0$ a.e. in Ω and $g \geq 0$ σ -a.e. on Γ . We shall also denote $F^+ = (f^+, g^+)$ and $F^- = (f^-, g^-)$ where $f^+ = \sup\{f, 0\}$ and $f^- = \sup\{-f, 0\}$. We have the following result.

Proposition 1.1. *The operator A_δ generates a strongly continuous analytic semigroup $(e^{tA_\delta})_{t \geq 0}$ on $\mathbb{X}^2(\overline{\Omega})$ which is also submarkovian. That is, the semigroup is positive and contractive on $\mathbb{X}^\infty(\overline{\Omega})$.*

Proof. Since the symmetric form \mathcal{E}_δ is closed, continuous, elliptic and $\mathbb{W}_\delta^{1,2}(\overline{\Omega})$ is dense in $\mathbb{X}^2(\overline{\Omega})$, we have that A_δ generates a strongly continuous and analytic semigroup $(e^{tA_\delta})_{t \geq 0}$ on $\mathbb{X}^2(\overline{\Omega})$. Next we show that the semigroup is positive. Let $U = (u, u_\Gamma) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$. Then $U^+ = (u^+, u_\Gamma^+) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ and a simple calculation gives $\mathcal{E}_\delta(U^+, U^-) = 0$. By [3, Theorem 1.3.2], this implies that the semigroup is positive. Next, for $0 \leq U = (u, u_\Gamma) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$, we let $U \wedge 1 = (u \wedge 1, u_\Gamma \wedge 1)$. It is also easy to see that for every $0 \leq U \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ we have that $U \wedge 1 \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ and a simple calculation gives $\mathcal{E}_\delta(U \wedge 1, U \wedge 1) \leq \mathcal{E}_\delta(U, U)$. By [3, Theorem 1.3.3], this implies that the semigroup is contractive on $\mathbb{X}^\infty(\overline{\Omega})$. We have shown that the semigroup is submarkovain and the proof is finished. \square

We adopt the following notion of solutions to the system (1.1).

Definition 1.2. *Let $g \in L^2(\Sigma_T)$, $F := (0, g)$ and $U_0 := (u_0, u_{0,\Gamma}) \in \mathbb{X}^2(\overline{\Omega})$.*

- (a) *A function u is said to be a strong solution of (1.1) if $U := (u, u|_\Gamma) \in W^{1,2}((0, T); \mathbb{X}^2(\overline{\Omega})) \cap L^2((0, T); D(A_\delta))$ and fulfills (1.1).*
- (b) *A function u is called a mild solution of (1.1) if $U := (u, u|_\Gamma) \in C([0, T]; \mathbb{X}^2(\overline{\Omega}))$ and satisfies*

$$U(\cdot, t) = e^{tA_\delta}U_0 + \int_0^t e^{(t-s)A_\delta}F(\cdot, s) ds \quad \text{in } \mathbb{X}^2(\overline{\Omega}), \quad t \in [0, T]. \quad (1.10)$$

Next, using a simple integration by parts, we have that the adjoint system associated to (1.1) is given by the following backward problem

$$\begin{cases} \partial_t \phi + \gamma \Delta \phi = 0 & \text{in } \Omega \times (0, T) = \Omega_T \\ \partial_t \phi_\Gamma + \delta \Delta_\Gamma \phi_\Gamma - \gamma \partial_\nu \phi - \beta \phi_\Gamma = 0 & \text{on } \Gamma \times (0, T) = \Sigma_T \\ (\phi, \phi_\Gamma)|_{t=T} = (\phi_T, \phi_{\Gamma,T}) & \text{in } \Omega \times \Gamma. \end{cases} \quad (1.11)$$

Here too, $\phi_{\Gamma,T}$ is not a priori the trace of ϕ_T since we did not assume that ϕ_T has a trace, but if ϕ_T has a well defined trace then the trace must coincide with $\phi_{\Gamma,T}$.

We notice that using the operator A_δ , we have that the system (1.1) can be rewritten as an abstract Cauchy problem

$$\partial_t U - A_\delta U = F \quad \text{in } \Omega_T \times \Sigma_T, \quad U(\cdot, 0) = (u_0, u_{\Gamma,0}) \quad \text{on } \Omega \times \Gamma, \quad (1.12)$$

where $U := (u, u_\Gamma)$ and $F := (0, g)$. Similarly, we have that the system (1.11) can be rewritten as

$$\partial_t \Phi + A_\delta \Phi = 0 \quad \text{in } \Omega_T \times \Sigma_T, \quad \Phi(\cdot, T) = (\phi_T, \phi_{\Gamma,T}) \quad \text{on } \Omega \times \Gamma, \quad (1.13)$$

where $\Phi := (\phi, \phi_\Gamma)$.

We have the following result of existence and uniqueness of solutions as a direct consequence of the generation result in Proposition 1.1.

Proposition 1.3. *The following assertions hold.*

- (a) For every $U_0 := (u_0, u_{\Gamma,0}) \in \mathbb{X}^2(\overline{\Omega})$ and $g \in L^2(\Sigma_T)$, the Cauchy problem (1.12), and hence the system (1.1), has a unique mild solution U given by (1.10). Moreover, there exists a constant $C > 0$ such that

$$\|U\|_{C([0,T];\mathbb{X}^2(\overline{\Omega}))} \leq C \left(\|U_0\|_{\mathbb{X}^2(\overline{\Omega})} + \|g\|_{L^2(\Sigma_T)} \right). \quad (1.14)$$

- (b) For every $U_0 := (u_0, u_{\Gamma,0}) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ and $g \in L^2(\Sigma_T)$, the Cauchy problem (1.12), and hence the system (1.1), has a unique strong solution U and there is a constant $C > 0$ such that

$$\|U\|_{W^{1,2}((0,T);\mathbb{X}^2(\overline{\Omega})) \cap L^2((0,T);D(A_\delta))} \leq C \left(\|U_0\|_{\mathbb{W}_\delta^{1,2}(\overline{\Omega})} + \|g\|_{L^2(\Sigma_T)} \right). \quad (1.15)$$

- (c) For every $\Phi_T := (\phi_T, \phi_{\Gamma,T}) \in \mathbb{X}^2(\overline{\Omega})$ (resp. $\Phi_T \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$) the Cauchy problem (1.13), and hence the backward system (1.11), has a unique mild solution Φ (resp. strong solution) given by

$$\Phi(\cdot, t) = e^{(T-t)A_\delta} \Phi_T \text{ in } \mathbb{X}^2(\overline{\Omega}), \quad t \in [0, T].$$

The generation of semigroup given in Proposition 1.1 and the proof of the existence and regularity of mild and strong solutions stated in Proposition 1.3 can be done by using the general well-posedness results of Cauchy problems associated with maximal monotone operators contained in the monograph by Brezis [2]. We also mention that Proposition 1.3 has been also completely proved in [13] by assuming that Ω is smooth.

Remark 1.4. It is easy to see that every strong solution is also a mild solution and a mild solution satisfying the regularity given in Definition 1.2(a) is also a strong solution. For more detail we refer to [2, 13] and their references.

1.4. The main result. In this (sub)section we state the results concerning the null controllability of the system (1.1). We start with a necessary and sufficient condition for the system to be null controllable.

Proposition 1.5. *The following assertions are equivalent.*

- (i) *The system (1.1) is null controllable in time $T > 0$.*
- (ii) *For every $(u_0, u_{\Gamma,0}) \in \mathbb{X}^2(\overline{\Omega})$, there exists a control function $g \in L^2(\Sigma_T)$ such that*

$$-\int_0^T \int_\Gamma g(x, t) \phi_\Gamma(x, t) \, d\sigma dt = \int_\Omega u_0(x) \phi(x, 0) \, dx + \int_\Gamma u_{\Gamma,0}(x) \phi_\Gamma(x, 0) \, d\sigma, \quad (1.16)$$

for every $(\phi_T, \phi_{\Gamma,T}) \in \mathbb{X}^2(\overline{\Omega})$, where ϕ is the unique mild solution of the backward system (1.11) with final data $(\phi_T, \phi_{\Gamma,T})$.

Proof. Let $g \in L^2(\Sigma_T)$ be arbitrary and u the solution of (1.1). Let $(\phi_T, \phi_{\Gamma,T}) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ and ϕ the strong solution of (1.11). Multiplying the first equation in (1.1) by ϕ and integrating by parts we get that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (\partial_t u - \gamma \Delta u) \phi \, dx dt = \int_\Omega (u(x, T) \phi_T(x) - u_0(x) \phi(x, 0)) \, dx - \int_0^T \int_\Omega u \partial_t \phi \, dx dt \\ &\quad - \int_0^T \int_\Omega \gamma u \Delta \phi \, dx dt - \gamma \int_0^T \int_\Gamma \partial_\nu u \phi \, d\sigma dt + \gamma \int_0^T \int_\Gamma \partial_\nu \phi u \, d\sigma dt. \end{aligned} \quad (1.17)$$

Multiplying the second equation in (1.1) by ϕ_Γ and integrating by parts we get that

$$\begin{aligned} \int_0^T \int_\Gamma g(x, t) \phi_\Gamma(x, t) \, d\sigma dt &= \int_\Gamma \left(u_\Gamma(x, T) \phi_{\Gamma, T}(x) - u_\Gamma(x, 0) \phi_\Gamma(x, 0) \right) d\sigma - \int_0^T \int_\Gamma u_\Gamma \partial_t \phi_\Gamma \, d\sigma dt \\ &\quad - \int_0^T \int_\Gamma u_\Gamma \delta \Delta_\Gamma \phi_\Gamma \, d\sigma dt + \int_0^T \int_\Gamma \left(\gamma \partial_\nu u + \beta u_\Gamma \right) \phi_\Gamma \, d\sigma dt. \end{aligned} \quad (1.18)$$

Adding (1.17) and (1.18) and using that ϕ is the solution of (1.11) we get that

$$\begin{aligned} \int_0^T \int_\Gamma g(x, t) \phi_\Gamma(x, t) \, d\sigma dt &= \int_\Omega \left(u(x, T) \phi_T(x) - u_0(x) \phi(x, 0) \right) dx \\ &\quad + \int_\Gamma \left(u_\Gamma(x, T) \phi_{\Gamma, T}(x) - u_{\Gamma, 0}(x) \phi_\Gamma(x, 0) \right) d\sigma \\ &= - \int_\Omega u_0(x) \phi(x, 0) - \int_\Gamma u_{\Gamma, 0}(x) \phi_\Gamma(x, 0) d\sigma \\ &\quad + \int_\Omega u(x, T) \phi_T(x) dx + \int_\Gamma u_\Gamma(x, T) \phi_{\Gamma, T}(x) d\sigma. \end{aligned} \quad (1.19)$$

By approximation, we have that the identity (1.19) also holds for every $(\phi_T, \phi_{\Gamma, T}) \in \mathbb{X}^2(\overline{\Omega})$ and ϕ the mild solution of (1.11).

Now assume that the system (1.1) is null controllable in time $T > 0$. Then by definition, $u(\cdot, T) = 0$ in Ω and $u_\Gamma(\cdot, T) = 0$ on Γ . Hence, it follows from (1.19) that the identity (1.16) holds and we have shown that (i) implies (ii). Finally assume that (1.16) holds. Then it follows from (1.19) again that

$$\int_\Omega u(x, T) \phi_T(x) dx + \int_\Gamma u_\Gamma(x, T) \phi_{\Gamma, T}(x) d\sigma = \langle (u(\cdot, T), u_\Gamma(\cdot, T)); (\phi_T, \phi_{\Gamma, T}) \rangle_{\mathbb{X}^2(\overline{\Omega})} = 0$$

for every $(\phi_T, \phi_{\Gamma, T}) \in \mathbb{X}^2(\overline{\Omega})$. Hence, $u(\cdot, T) = 0$ in Ω , $u_\Gamma(\cdot, T) = 0$ on Γ and we have shown that the system (1.1) is null controllable in time $T > 0$. The proof is finished. \square

The following theorem is the main result of the paper.

Theorem 1.6. *For every $T > 0$ and $(u_0, u_{\Gamma, 0}) \in \mathbb{X}^2(\overline{\Omega})$, there exists a control function $g \in L^2(\Sigma_T)$ such that the unique mild solution u of (1.1) satisfies $u(\cdot, T) = 0$ in Ω and $u_\Gamma(\cdot, T) = 0$ on Γ .*

2. PROOF OF THE MAIN RESULT

In this section we give the proof of Theorem 1.6. To do this we need some important intermediate results. We start with the so called Carleman estimate.

2.1. The Carleman estimates. The following theorem gives a suitable Carleman type estimate for solutions of the backward system (1.11). It is one of the main tool needed in the proof of the main result. The proof of the Carleman estimates given here uses some ideas of the corresponding result in the case of the interior null controllability of the system studied in [13]. The weight functions used are the same as the ones in [6] for the Dirichlet boundary condition.

Theorem 2.1. *Let $T > 0$ and $m > 1$ be real numbers. Given a positive parameter λ , we define the weight function α on $\overline{\Omega} \times (0, T)$ by*

$$\alpha(x, t) = \theta(t) p(x) := \frac{1}{t(T-t)} \left(e^{2\lambda m \|\eta\|_\infty} - e^{\lambda(m \|\eta\|_\infty + \eta(x))} \right), \quad (2.1)$$

where $\eta \in C^2(\overline{\Omega})$ is such that $\eta > 0$ in Ω and $\eta = 0$ on Γ . Then, there exists a constant $C > 0$ and $\lambda_0, R_0 > 1$ such that for all $\lambda > \lambda_0$ and $R > R_0$ the strong solution ϕ of the backward system (1.11) satisfies

$$\begin{aligned} & \lambda^3 R^2 \int_{\Omega_T} \theta^3 \xi^3 e^{-2R\alpha} \phi^2 dxdt + \lambda \int_{\Omega_T} \theta \xi e^{-2R\alpha} |\nabla \phi|^2 dxdt \\ & + \lambda^2 R^2 \int_{\Sigma_T} \theta^3 \xi^3 e^{-2R\alpha} \phi_\Gamma^2 d\sigma dt \leq C \int_{\Sigma_T} \theta \xi e^{-2R\alpha} |\partial_t \phi_\Gamma + \delta \Delta_\Gamma \phi_\Gamma - \gamma \partial_\nu \phi|^2 d\sigma dt, \end{aligned} \quad (2.2)$$

where, for simplicity of the notation, we have set

$$\xi(x) := e^{\lambda(m\|\eta\|_\infty + \eta(x))}. \quad (2.3)$$

Proof. We give the proof in several steps. Throughout the proof C will denote a generic constant which is independent of λ , R and ϕ . This constant may vary even from line to line.

Step 1. An auxiliary problem. For any strong solution ϕ of the adjoint system (1.11) and for any fixed $R > 0$, we define

$$z(x, t) := \phi(x, t) e^{-R\alpha(x, t)}. \quad (2.4)$$

We notice that

$$\begin{cases} z(\cdot, 0) := \lim_{t \rightarrow 0} z(\cdot, t) = 0 = z(\cdot, T) := \lim_{t \rightarrow T} z(\cdot, t) & \text{in } \Omega \\ z_\Gamma(\cdot, 0) := \lim_{t \rightarrow 0} z_\Gamma(\cdot, t) = 0 = z_\Gamma(\cdot, T) := \lim_{t \rightarrow T} z_\Gamma(\cdot, t) & \text{on } \Gamma \end{cases} \quad (2.5)$$

The parameter R is meant to be large. Plugging the function $\phi(x, t) = z(x, t) e^{R\alpha(x, t)}$ in the system (1.11) and using (2.5), we obtain that z satisfies the following system

$$\begin{cases} \partial_t z + \gamma \Delta z + 2R\gamma \nabla \alpha \cdot \nabla z + (R\gamma \Delta \alpha + R\alpha_t)z + R^2 |\nabla \alpha|^2 z = 0 & \text{in } \Omega_T \\ \partial_t z_\Gamma + \delta \Delta_\Gamma z_\Gamma - \gamma \partial_\nu z - (\beta - R\alpha_t + R\gamma \partial_\nu \alpha) z_\Gamma = 0 & \text{on } \Sigma_T \\ z(\cdot, 0) = z(\cdot, T) = 0 & \text{in } \Omega \\ z_\Gamma(\cdot, 0) = z_\Gamma(\cdot, T) = 0 & \text{on } \Gamma \end{cases} \quad (2.6)$$

where $\alpha_t := \partial_t \alpha$. Next, expanding the spatial derivatives of α by using the chain rule, we obtain

$$\nabla \alpha = -\lambda \theta \xi \nabla \eta \quad \text{and} \quad \Delta \alpha = -\lambda \theta \xi \Delta \eta - \lambda^2 \theta \xi |\nabla \eta|^2,$$

where we recall that we have used the notation (2.3). Using the above expressions, the system (2.6) can be rewritten as

$$\begin{cases} M_1 + M_2 = f & \text{in } \Omega_T \\ N_1 + N_2 = h & \text{on } \Sigma_T \end{cases} \quad (2.7)$$

where

$$\begin{cases} M_1 &:= -2\lambda^2 R \gamma \theta \xi |\nabla \eta|^2 z - 2\lambda R \gamma \theta \xi (\nabla \eta \cdot \nabla z) + \partial_t z := M_{1,1} + M_{1,2} + M_{1,3} \\ M_2 &:= \lambda^2 R^2 \gamma \theta^2 \xi^2 |\nabla \eta|^2 z + \gamma \Delta z + R\alpha_t z := M_{2,1} + M_{2,2} + M_{2,3} \\ N_1 &:= \partial_t z_\Gamma + \lambda R \gamma \theta \xi \partial_\nu \eta z_\Gamma := N_{1,1} + N_{1,2} \\ N_2 &:= \delta \Delta_\Gamma z_\Gamma + R\alpha_t z_\Gamma - \gamma \partial_\nu z := N_{2,1} + N_{2,2} + N_{2,3} \\ f &:= \lambda R \gamma \theta \xi \Delta \eta z - \lambda^2 R \gamma \theta \xi |\nabla \eta|^2 z \\ h &:= \beta z_\Gamma. \end{cases}$$

Applying the respective L^2 -norms to the terms in (2.7) we get that

$$\begin{aligned} \|f\|_{L^2(\Omega_T)}^2 + \|h\|_{L^2(\Sigma_T)}^2 &= \|M_1\|_{L^2(\Omega_T)}^2 + \|M_2\|_{L^2(\Omega_T)}^2 + \|N_1\|_{L^2(\Sigma_T)}^2 + \|N_2\|_{L^2(\Sigma_T)}^2 \\ &\quad + 2\langle M_1, M_2 \rangle_{L^2(\Omega_T)} + 2\langle N_1, N_2 \rangle_{L^2(\Sigma_T)}, \end{aligned} \quad (2.8)$$

which clearly implies that

$$\begin{aligned} 2\langle M_1, M_2 \rangle_{L^2(\Omega_T)} - \|f\|_{L^2(\Omega_T)}^2 + 2\langle N_1, N_2 \rangle_{L^2(\Sigma_T)} - \|h\|_{L^2(\Sigma_T)}^2 \\ = - \sum_{i=1}^2 \left(\|M_i\|_{L^2(\Omega_T)}^2 + \|N_i\|_{L^2(\Sigma_T)}^2 \right) \leq 0. \end{aligned} \quad (2.9)$$

Step 2. Estimate from below of the terms defined on Ω_T in (2.9). We compute and estimate from below the scalar product $\langle M_1, M_2 \rangle_{L^2(\Omega_T)} - \|f\|_{L^2(\Omega_T)}^2$.

Step 2.1. Estimate from below of $\langle M_1, M_{2,1} \rangle_{L^2(\Omega_T)}$. Calculating we have that

$$\langle M_{1,1}, M_{2,1} \rangle_{L^2(\Omega_T)} = -2\lambda^4 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 |\nabla \eta|^4 z^2 dx dt. \quad (2.10)$$

A simple calculation and an integrating by parts yield

$$\begin{aligned} &\langle M_{1,2}, M_{2,1} \rangle_{L^2(\Omega_T)} \\ &= -\lambda^3 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 |\nabla \eta|^2 (\nabla \eta \cdot \nabla(z^2)) dx dt \\ &= -\lambda^3 R^3 \gamma^2 \int_{\Sigma_T} \theta^3 \xi^3 |\nabla \eta|^2 \partial_\nu \eta z^2 d\sigma dt + \lambda^3 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \operatorname{div}(\xi^3 |\nabla \eta|^2 \nabla \eta) z^2 dx dt \\ &= \mathbf{BT}_1 + 3\lambda^4 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 |\nabla \eta|^4 z^2 dx dt + \lambda^3 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 \Delta \eta |\nabla \eta|^2 z^2 dx dt \\ &\quad + \lambda^3 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 (\nabla \eta \cdot \nabla(|\nabla \eta|^2)) z^2 dx dt, \end{aligned} \quad (2.11)$$

where we have set the boundary term

$$\mathbf{BT}_1 := -\lambda^3 R^3 \gamma^2 \int_{\Sigma_T} \theta^3 \xi^3 |\nabla \eta|^2 \partial_\nu \eta z^2 d\sigma dt.$$

Adding (2.10) and (2.11) and using that

$$\Delta \eta \geq -|\Delta \eta| \quad \text{and} \quad \nabla \eta \cdot \nabla(|\nabla \eta|^2) \geq -|\nabla \eta| |\nabla(|\nabla \eta|^2)|,$$

we get that

$$\begin{aligned} &\langle M_{1,1} + M_{1,2}, M_{2,1} \rangle_{L^2(\Omega_T)} \\ &\geq \mathbf{BT}_1 + \lambda^4 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 \left(|\nabla \eta|^4 - \frac{1}{\lambda} |\Delta \eta| |\nabla \eta|^2 \right) z^2 dx dt \\ &\quad - \lambda^3 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 |\nabla \eta| |\nabla(|\nabla \eta|^2)| z^2 dx dt \\ &= \mathbf{BT}_1 + \lambda^4 R^3 \gamma^2 \int_{\Omega_T} \theta^3 \xi^3 \left(|\nabla \eta|^4 - \frac{1}{\lambda} |\Delta \eta| |\nabla \eta|^2 - \frac{1}{\lambda} |\nabla \eta| |\nabla(|\nabla \eta|^2)| \right) z^2 dx dt. \end{aligned} \quad (2.12)$$

Let

$$\lambda \geq \max \left\{ \frac{2|\Delta \eta|}{|\nabla \eta|^2}, \frac{4|\nabla(|\nabla \eta|^2)|}{|\nabla \eta|^3} \right\}. \quad (2.13)$$

Then

$$|\nabla\eta|^4 - \frac{1}{\lambda}|\Delta\eta||\nabla\eta|^2 \geq \frac{1}{2}|\nabla\eta|^4 \quad \text{and} \quad \frac{1}{2}|\nabla\eta|^4 - \frac{1}{\lambda}|\nabla\eta||\nabla(|\nabla\eta|^2)| \geq \frac{1}{4}|\nabla\eta|^4. \quad (2.14)$$

It follows from (2.14) that if λ satisfies (2.13), then

$$|\nabla\eta|^4 - \frac{1}{\lambda}|\Delta\eta||\nabla\eta|^2 - \frac{1}{\lambda}|\nabla\eta||\nabla(|\nabla\eta|^2)| \geq \frac{1}{4}|\nabla\eta|^4. \quad (2.15)$$

Using (2.15) we get from (2.12) that there exists a constant $C > 0$ (depending only on η and γ) such that if λ satisfies (2.13), then

$$\langle M_{1,1} + M_{1,2}, M_{2,1} \rangle_{L^2(\Omega_T)} \geq \mathbf{BT}_1 + \lambda^4 R^3 C \int_{\Omega_T} \theta^3 \xi^3 z^2 dx dt. \quad (2.16)$$

Moreover, we have that there is a constant $C > 0$ (depending only on η and γ) such that

$$\langle M_{1,3}, M_{2,1} \rangle_{L^2(\Omega_T)} = -\lambda^2 R^2 \gamma \int_{\Omega_T} \theta \theta_t \xi^2 |\nabla\eta|^2 z^2 dx dt \geq -\lambda^2 R^2 C \int_{\Omega_T} \theta^3 \xi^3 z^2 dx dt, \quad (2.17)$$

where we have used the fact that there is a constant $C > 0$ such that $|\theta_t| \leq C\theta^2$. It follows from (2.16) and (2.17) that if λ satisfies (2.13), then

$$\langle M_1, M_{2,1} \rangle_{L^2(\Omega_T)} \geq \mathbf{BT}_1 + \lambda^4 R^3 C \int_{\Omega_T} \theta^3 \xi^3 \left(1 - \frac{1}{R\lambda^2}\right) z^2 dx dt. \quad (2.18)$$

If $\lambda^2 \geq \frac{2}{R}$, then $\left(1 - \frac{1}{R\lambda^2}\right) \geq \frac{1}{2}$. Thus it follows from (2.18) that there exists a constant $C > 0$ such that if

$$\lambda \geq \max \left\{ \frac{2|\Delta\eta|}{|\nabla\eta|^2}, \frac{4|\nabla(|\nabla\eta|)|}{|\nabla\eta|^3}, \sqrt{\frac{2}{R}} \right\},$$

then

$$\langle M_1, M_{2,1} \rangle_{L^2(\Omega_T)} \geq \mathbf{BT}_1 + \lambda^4 R^3 C \int_{\Omega_T} \theta^3 \xi^3 z^2 dx dt. \quad (2.19)$$

Step 2.2. Estimate from below of $\langle M_1, M_{2,2} \rangle_{L^2(\Omega_T)}$. Integrating by parts and using the fact that

$$\nabla\xi = \lambda\xi\nabla\eta, \quad (2.20)$$

we get that

$$\begin{aligned} \langle M_{1,1}, M_{2,2} \rangle_{L^2(\Omega_T)} &= -2\lambda^2 R \gamma^2 \int_{\Sigma_T} \theta \xi |\nabla\eta|^2 z \partial_\nu z d\sigma dt \\ &\quad + 2\lambda^2 R \gamma^2 \int_{\Omega_T} \theta \xi |\nabla\eta|^2 |\nabla z|^2 dx dt \\ &\quad + 2\lambda^2 R \gamma^2 \int_{\Omega_T} \theta \xi (\nabla(|\nabla\eta|^2) \cdot \nabla z) z dx dt \\ &\quad + 2\lambda^3 R \gamma^2 \int_{\Omega_T} \theta \xi |\nabla\eta|^2 (\nabla\eta \cdot \nabla z) z dx dt \\ &=: \mathbf{BT}_2 + \mathbf{DT}_1 + \mathbf{DT}_2 + \mathbf{DT}_3. \end{aligned} \quad (2.21)$$

Now applying the Young inequality, we get that there exists a constant $C > 0$ (depending only on η and γ) such that

$$\begin{aligned} |\mathbf{DT}_2| &= \left| 2\lambda^2 R\gamma^2 \int_{\Omega_T} \theta \xi (\nabla(|\nabla\eta|^2) \cdot \nabla z) z \, dxdt \right| \\ &\leq \lambda^4 RC \int_{\Omega_T} \theta \xi z^2 \, dxdt + RC \int_{\Omega_T} \theta \xi |\nabla z|^2 \, dxdt \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} |\mathbf{DT}_3| &= \left| 2\lambda^3 R\gamma^2 \int_{\Omega_T} \theta \xi |\nabla\eta|^2 (\nabla\eta \cdot \nabla z) z \, dxdt \right| \\ &\leq \lambda^4 R^2 C \int_{\Omega_T} \theta^2 \xi^2 z^2 \, dxdt + \lambda^2 C \int_{\Omega_T} |\nabla z|^2 \, dxdt. \end{aligned} \quad (2.23)$$

Using (2.22) and (2.23), we get from (2.21) that there exists a constant $C > 0$ such that

$$\begin{aligned} \langle M_{1,1}, M_{2,2} \rangle_{L^2(\Omega_T)} &\geq \mathbf{DT}_1 - \lambda^4 R^2 C \int_{\Omega_T} \left(1 + \frac{1}{\lambda^2 R \theta \xi} \right) \theta^2 \xi^2 z^2 \, dxdt \\ &\quad - C \int_{\Omega_T} (R\theta\xi + \lambda^2) |\nabla z|^2 \, dxdt + \mathbf{BT}_2. \end{aligned} \quad (2.24)$$

Furthermore, we have that

$$\begin{aligned} &\langle M_{1,2}, M_{2,2} \rangle_{L^2(\Omega_T)} \\ &= -2\lambda R\gamma^2 \int_{\Sigma_T} \theta \xi (\nabla\eta \cdot \nabla z) \partial_\nu z \, d\sigma dt + 2\lambda R\gamma^2 \int_{\Omega_T} \theta \nabla (\xi (\nabla\eta \cdot \nabla z)) \cdot \nabla z \, dxdt \\ &= -2\lambda R\gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\partial_\nu z|^2 \, d\sigma dt + 2\lambda R\gamma^2 \int_{\Omega_T} \theta \xi \nabla (\nabla\eta \cdot \nabla z) \cdot \nabla z \, dxdt \\ &\quad + 2\lambda R\gamma^2 \int_{\Omega_T} \theta (\nabla\eta \cdot \nabla z) (\nabla\eta \cdot \nabla z) \, dxdt \\ &= \mathbf{BT}_3 + 2\lambda R\gamma^2 \int_{\Omega_T} \theta \xi D^2\eta (\nabla z, \nabla z) \, dxdt + \lambda R\gamma^2 \int_{\Omega_T} \theta \xi (\nabla\eta \cdot \nabla(|\nabla z|^2)) \, dxdt \\ &\quad + 2\lambda^2 R\gamma^2 \int_{\Omega_T} \theta \xi (\nabla\eta \cdot \nabla z)^2 \, dxdt \\ &= \mathbf{BT}_3 + \mathbf{DT}_4 + \mathbf{DT}_5 + \mathbf{DT}_6. \end{aligned} \quad (2.25)$$

In the above expression, in \mathbf{DT}_4 we have introduced the notation

$$D^2\eta(\zeta, \zeta) := \sum_{i,j=1}^N \partial_{x_i x_j} \eta \zeta_i \zeta_j, \quad \forall \zeta \in \mathbb{R}^N.$$

Moreover we have that there exists a constants $C > 0$ such that

$$\begin{aligned} \mathbf{DT}_4 &= 2\lambda R\gamma^2 \int_{\Omega_T} \theta \xi D^2\eta (\nabla z, \nabla z) \, dxdt \\ &\geq -\lambda RC \int_{\Omega_T} \theta \xi |\nabla z|^2 \, dxdt \end{aligned} \quad (2.26)$$

and

$$\begin{aligned}
\mathbf{DT}_5 &= \lambda R \gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\nabla z|^2 d\sigma dt - \lambda R \gamma^2 \int_{\Omega_T} \theta \operatorname{div}(\xi \nabla \eta) |\nabla z|^2 dx dt \\
&= \mathbf{BT}_4 - \lambda R \gamma^2 \int_{\Omega_T} \theta \xi \Delta \eta |\nabla z|^2 dx dt - \lambda R \gamma^2 \int_{\Omega_T} \theta (\nabla \xi \cdot \nabla \eta) |\nabla z|^2 dx dt \\
&\geq \mathbf{BT}_4 - \lambda R C \int_{\Omega_T} \theta \xi |\nabla z|^2 dx dt - \lambda^2 R \gamma^2 \int_{\Omega_T} \theta \xi |\nabla \eta|^2 |\nabla z|^2 dx dt,
\end{aligned} \tag{2.27}$$

where we have also used (2.20). In addition we have that there exists a constant $C > 0$ such that

$$\mathbf{DT}_6 = 2\lambda^2 R \gamma^2 \int_{\Omega_T} \theta \xi (\nabla \eta \cdot \nabla z)^2 dx dt \geq 2\lambda^2 R C \int_{\Omega_T} \theta \xi |\nabla z|^2 dx dt. \tag{2.28}$$

Since ∇z vanishes at $t = 0$ and at $t = T$, then integrating by parts, we get that

$$\langle M_{1,3}, M_{2,2} \rangle_{L^2(\Omega_T)} = \gamma \int_{\Sigma_T} z_t \partial_\nu z d\sigma dt =: \mathbf{BT}_5. \tag{2.29}$$

Combining (2.24), (2.25), (2.26), (2.27), (2.28) and (2.29) we get that for λ large enough,

$$\begin{aligned}
\langle M_1, M_{2,2} \rangle_{L^2(\Omega_T)} &\geq \mathbf{BT}_1 + \mathbf{BT}_2 + \mathbf{BT}_3 + \mathbf{BT}_4 + \mathbf{BT}_5 + \lambda^2 R C \int_{\Omega_T} \theta \xi |\nabla z|^2 dx dt \\
&\quad - \lambda^4 R^2 C \int_{\Omega_T} \theta^2 \xi^2 z^2 dx dt,
\end{aligned} \tag{2.30}$$

for some constant $C > 0$ depending only on η and γ .

Step 2.3. Estimate from below of $\langle M_1, M_{2,3} \rangle_{L^2(\Omega_T)}$. First, we notice that there exist two constants $\varsigma_1 > 0$ and $\varsigma_2 > 0$ such that

$$|\alpha_t| \leq \varsigma_1 \theta^2 \xi^2 \quad \text{and} \quad |\alpha_{tt}| \leq \varsigma_2 \theta^3 \xi^3. \tag{2.31}$$

Next, using (2.31), we get that there exists a constant $C > 0$ such that if λ is large enough, then

$$\begin{aligned}
\langle M_{1,1}, M_{2,3} \rangle_{L^2(\Omega_T)} &= -2\lambda^2 R^2 \gamma \int_{\Omega_T} \theta \xi |\nabla \eta|^2 \alpha_t z^2 dx dt \\
&\geq -\lambda^2 R^2 C \int_{\Omega_T} \theta^3 \xi^3 z^2 dx dt.
\end{aligned} \tag{2.32}$$

Calculating and integrating by parts we get that

$$\begin{aligned}
& \langle M_{1,2}, M_{2,3} \rangle_{L^2(\Omega_T)} \\
&= -\lambda R^2 \gamma \int_{\Omega_T} \theta \xi \alpha_t (\nabla \eta \cdot \nabla (z^2)) \, dxdt \\
&= -\lambda R^2 \gamma \int_{\Sigma_T} \theta \xi \alpha_t \partial_\nu \eta z^2 \, d\sigma dt + \lambda R^2 \gamma \int_{\Omega_T} \theta \operatorname{div}(\xi \alpha_t \nabla \eta) z^2 \, dxdt \\
&= \mathbf{BT}_6 + \lambda R^2 \gamma \int_{\Omega_T} \theta \xi \alpha_t \Delta \eta z^2 \, dxdt + \lambda R^2 \gamma \int_{\Omega_T} \theta \xi (\nabla \alpha_t \cdot \nabla \eta) z^2 \, dxdt \\
&\quad + \lambda R^2 \gamma \int_{\Omega_T} \theta \alpha_t (\nabla \xi \cdot \nabla \eta) z^2 \, dxdt \\
&= \mathbf{BT}_6 + \lambda R^2 \gamma \int_{\Omega_T} \theta \xi \alpha_t \Delta \eta z^2 \, dxdt + \lambda R^2 \gamma \int_{\Omega_T} \theta \xi (\nabla \alpha_t \cdot \nabla \eta) z^2 \, dxdt \\
&\quad + \lambda^2 R^2 \gamma \int_{\Omega_T} \theta \alpha_t (\nabla \eta \cdot \nabla \eta) z^2 \, dxdt \\
&\geq \mathbf{BT}_6 - \lambda^2 R^2 C \int_{\Omega_T} \theta^3 \xi^3 z^2 \, dxdt,
\end{aligned} \tag{2.33}$$

where we have set

$$\mathbf{BT}_6 := -\lambda R^2 \gamma \int_{\Sigma_T} \theta \xi \alpha_t \partial_\nu \eta z^2 \, d\sigma dt.$$

In addition we have that there exists a constant $C > 0$ such that

$$\langle M_{1,3}, M_{2,3} \rangle_{L^2(\Omega_T)} = \frac{R}{2} \int_{\Omega_T} \alpha_t (z^2)_t \, dxdt = -\frac{R}{2} \int_{\Omega_T} \alpha_{tt} z^2 \, dxdt \geq -RC \int_{\Omega_T} \theta^3 \xi^3 z^2 \, dxdt. \tag{2.34}$$

Combining (2.32), (2.33) and (2.34), we get that there exists a constant $C > 0$ such that for λ and R large enough, we have

$$\langle M_1, M_{2,3} \rangle_{L^2(\Omega_T)} \geq \mathbf{BT}_6 - \lambda^2 R^2 C \int_{\Omega_T} \theta^3 \xi^3 z^2 \, dxdt. \tag{2.35}$$

Finally, it follows from (2.19), (2.30) and (2.35) that there is a constant $C > 0$ such that for λ and R large enough, we have

$$\begin{aligned}
\langle M_1, M_2 \rangle_{L^2(\Omega_T)} &\geq \lambda^4 R^3 C \int_{\Omega_T} \theta^3 \xi^3 z^2 \, dxdt + \lambda^2 R C \int_{\Omega_T} \theta \xi |\nabla z|^2 \, dxdt \\
&\quad + \mathbf{BT}_1 + \mathbf{BT}_2 + \mathbf{BT}_3 + \mathbf{BT}_4 + \mathbf{BT}_5 + \mathbf{BT}_6.
\end{aligned} \tag{2.36}$$

Step 2.4. Estimate from below of $-\|f\|_{L^2(\Omega_T)}^2$. We have that there exists a constant $C > 0$ such that

$$\begin{aligned}
-\|f\|_{L^2(\Omega_T)}^2 &= -\|\lambda R \gamma \theta \xi \Delta \eta z - \lambda^2 R \gamma \theta \xi |\nabla \eta|^2 z\|_{L^2(Q)}^2 \\
&\geq -\lambda^2 R^2 \gamma^2 C \int_{\Omega_T} \theta^2 \xi^2 |\Delta \eta|^2 z^2 \, dxdt - \lambda^4 R^2 \gamma^2 C \int_{\Omega_T} \theta^2 \xi^2 |\nabla \eta|^4 z^2 \, dxdt \\
&\geq -\lambda^2 R^2 C \int_{\Omega_T} \theta^2 \xi^2 z^2 \, dxdt - \lambda^4 R^2 C \int_{\Omega_T} \theta^2 \xi^2 z^2 \, dxdt.
\end{aligned} \tag{2.37}$$

It follows from (2.36) and (2.37), that for λ and R large enough, all these terms can be absorbed and we get that there exists a constant $C > 0$ such that

$$\begin{aligned} 2\langle M_1, M_2 \rangle_{L^2(\Omega_T)} - \|f\|_{L^2(Q)}^2 &\geq \lambda^4 R^3 C \int_{\Omega_T} \theta^3 \xi^3 z^2 dxdt + \lambda^2 R C \int_{\Omega_T} \theta \xi |\nabla z|^2 dxdt \\ &\quad + \mathbf{BT}_1 + \mathbf{BT}_2 + \mathbf{BT}_3 + \mathbf{BT}_4 + \mathbf{BT}_5 + \mathbf{BT}_6. \end{aligned} \quad (2.38)$$

Step 3. Estimate of the boundary terms. Let us now compute the boundary terms. Integrating by parts, we get that

$$\begin{aligned} \langle N_{1,1}, N_{2,1} \rangle_{L^2(\Sigma_T)} &= \delta \int_{\Sigma_T} \partial_t z_\Gamma \Delta_\Gamma z_\Gamma d\sigma dt = -\delta \int_{\Sigma_T} \nabla_\Gamma (\partial_t z_\Gamma) \cdot \nabla_\Gamma z_\Gamma d\sigma dt \\ &= -\frac{\delta}{2} \int_{\Sigma_T} \partial_t (|\nabla_\Gamma z_\Gamma|^2) d\sigma dt = 0, \\ \langle N_{1,2}, N_{2,1} \rangle_{L^2(\Sigma_T)} &= \lambda R \gamma \delta \int_{\Sigma_T} \theta \xi \partial_\nu \eta z_\Gamma \Delta_\Gamma z_\Gamma d\sigma dt \\ &= -\lambda R \gamma \delta \int_{\Sigma_T} \theta \xi z_\Gamma \nabla_\Gamma (\partial_\nu \eta) \cdot \nabla_\Gamma z_\Gamma d\sigma dt - \lambda R \gamma \delta \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\nabla_\Gamma z_\Gamma|^2 d\sigma dt \\ \langle N_{1,1}, N_{2,2} \rangle_{L^2(\Sigma_T)} &= \frac{R}{2} \int_{\Sigma_T} \alpha_t \partial_t (z_\Gamma^2) d\sigma dt = -\frac{R}{2} \int_{\Sigma_T} \alpha_{tt} z_\Gamma^2 d\sigma dt \\ \langle N_{1,2}, N_{2,2} \rangle_{L^2(\Sigma_T)} &= \lambda R^2 \gamma \int_{\Sigma_T} \theta \xi \alpha_t \partial_\nu \eta z_\Gamma^2 d\sigma dt \\ \langle N_{1,1}, N_{2,3} \rangle_{L^2(\Sigma_T)} &= -\gamma \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \\ \langle N_{1,1}, N_{2,3} \rangle_{L^2(\Sigma_T)} &= -\lambda R \gamma \int_{\Sigma_T} \theta \xi \partial_\nu \eta z_\Gamma \partial_\nu z d\sigma dt. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} &\sum_{i=1}^6 \mathbf{BT}_i + 2\langle N_1, N_2 \rangle_{L^2(\Sigma_T)} - \|h\|_{L^2(\Sigma_T)}^2 \\ &= -\lambda^3 R^3 \gamma^2 \int_{\Sigma_T} \theta^3 \xi^3 |\nabla \eta|^2 \partial_\nu \eta z_\Gamma^2 d\sigma dt - 2\lambda^2 R \gamma^2 \int_{\Sigma_T} \theta \xi |\nabla \eta|^2 z_\Gamma \partial_\nu z d\sigma dt \\ &\quad + \lambda R^2 \gamma \int_{\Sigma_T} \theta \xi \alpha_t \partial_\nu \eta z_\Gamma^2 d\sigma dt - R \int_{\Sigma_T} \alpha_{tt} z_\Gamma^2 d\sigma dt - \int_{\Sigma_T} \beta^2 z_\Gamma^2 d\sigma dt \\ &\quad - 2\lambda R \gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta z_\Gamma \partial_\nu z d\sigma dt - \gamma \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \\ &\quad - 2\lambda R \gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\partial_\nu z|^2 d\sigma dt + \lambda R \gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\nabla z|^2 d\sigma dt \\ &\quad - 2\lambda R \gamma \delta \int_{\Sigma_T} \theta \xi z_\Gamma \nabla_\Gamma (\partial_\nu \eta) \cdot \nabla_\Gamma z_\Gamma d\sigma dt - 2\lambda R \gamma \delta \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\nabla_\Gamma z_\Gamma|^2 d\sigma dt =: J. \end{aligned}$$

Moreover,

$$\begin{aligned} -2\lambda R\gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\partial_\nu z|^2 d\sigma dt &= -2\lambda R\gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta (\nu \cdot \nabla z)^2 d\sigma dt \\ &\geq -2\lambda R\gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\nabla z|^2 d\sigma dt, \end{aligned}$$

where we notice that

$$|\nabla z|^2|_\Gamma = |\nabla_\Gamma z|^2 + |\partial_\nu z|^2.$$

Hence, we obtain a first estimate for the expression J , namely

$$\begin{aligned} J &\geq -\lambda^3 R^3 \gamma^2 \int_{\Sigma_T} \theta^3 \xi^3 |\nabla \eta|^2 \partial_\nu \eta z_\Gamma^2 d\sigma dt - 2\lambda^2 R \gamma^2 \int_{\Sigma_T} \theta \xi |\nabla \eta|^2 z_\Gamma \partial_\nu z d\sigma dt \\ &\quad + \lambda R^2 \gamma \int_{\Sigma_T} \theta \xi \alpha_t \partial_\nu \eta z_\Gamma^2 d\sigma dt - R \int_{\Sigma_T} \alpha_{tt} z_\Gamma^2 d\sigma dt - \int_{\Sigma_T} \beta^2 z_\Gamma^2 d\sigma dt \\ &\quad - 2\lambda R \gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta z_\Gamma \partial_\nu z d\sigma dt - \gamma \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt - 2\lambda R \gamma \delta \int_{\Sigma_T} \theta \xi z_\Gamma \nabla_\Gamma (\partial_\nu \eta) \cdot \nabla_\Gamma z_\Gamma d\sigma dt \\ &\quad - 2\lambda R \gamma \delta \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\nabla z|^2 d\sigma dt - \lambda R \gamma^2 \int_{\Sigma_T} \theta \xi \partial_\nu \eta |\nabla z|^2 d\sigma dt. \end{aligned} \quad (2.39)$$

We mention that

$$\nabla_\Gamma \eta = 0, \quad |\nabla \eta| = |\partial_\nu \eta|, \quad \partial_\nu \eta \leq -C < 0 \quad \text{on } \Gamma, \quad (2.40)$$

for some constant $C > 0$. Now since $\partial_\nu \eta|_\Gamma < 0$ (by (2.40)), we have that the last two terms in the right-hand side of (2.39) are positive and we can ignore them. Moreover, we recall that there exists a constant $C > 0$ such that $|\nabla \eta| \leq C$ and that $|\alpha_t| \leq C\theta^2 \xi^2$. Therefore, from (2.39) we obtain

$$\begin{aligned} J &\geq \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{\lambda^2 R}\right) z_\Gamma^2 d\sigma dt - \lambda^2 R C \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - R \int_{\Sigma_T} \alpha_{tt} z_\Gamma^2 d\sigma dt \\ &\quad - \int_{\Sigma_T} \theta^3 \xi^3 (\theta \xi)^{-3} \beta^2 z_\Gamma^2 d\sigma dt - 2\lambda R C \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - \gamma \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \\ &\quad - 2\lambda R \gamma \delta \int_{\Sigma_T} \theta \xi z_\Gamma \nabla_\Gamma (\partial_\nu \eta) \cdot \nabla_\Gamma z_\Gamma d\sigma dt. \end{aligned} \quad (2.41)$$

In order to treat the last integral in the right-hand side of (2.41), we recall from Section 1.2 that $\|\cdot\|_{L^2(\Gamma)} + \|\nabla_\Gamma \cdot\|_{L^2(\Gamma)}$ defines an equivalent norm on $W^{1,2}(\Gamma)$. Hence, the interpolation inequality (1.4) yields

$$\|\nabla_\Gamma z_\Gamma\|_{L^2(\Gamma)}^2 \leq C \|z_\Gamma\|_{L^2(\Gamma)} \|z_\Gamma\|_{D(\Delta_\Gamma)}.$$

Therefore, we have

$$\begin{aligned}
\left| \int_{\Sigma_T} \delta \theta \xi z_\Gamma \nabla_\Gamma(\partial_\nu \eta) \cdot \nabla_\Gamma z_\Gamma d\sigma dt \right| &\leq \int_{\Sigma_T} \theta \xi |\nabla_\Gamma(\partial_\nu \eta)| |\delta \nabla_\Gamma z_\Gamma| |z_\Gamma| d\sigma dt \leq C \int_{\Sigma_T} \theta \xi |\delta \nabla_\Gamma z_\Gamma| |z_\Gamma| d\sigma dt \\
&\leq C \int_0^T \theta \xi \delta^2 \|\nabla_\Gamma z_\Gamma\|_{L^2(\Gamma)}^2 dt + C \int_{\Sigma_T} \theta \xi z_\Gamma^2 d\sigma dt \\
&\leq C \int_0^T \theta \xi \delta^2 \|z_\Gamma\|_{D(\Delta_\Gamma)}^2 dt + C \int_{\Sigma_T} \theta \xi z_\Gamma^2 d\sigma dt \\
&\leq C \int_{\Sigma_T} \theta \xi |\delta \Delta_\Gamma z_\Gamma|^2 d\sigma dt + C \int_{\Sigma_T} \theta \xi z_\Gamma^2 d\sigma dt.
\end{aligned} \tag{2.42}$$

Plugging (2.42) in (2.41) we get

$$\begin{aligned}
J &\geq \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{\lambda^2 R}\right) z_\Gamma^2 d\sigma dt - \lambda^2 R C \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - R \int_{\Sigma_T} \alpha_{tt} z_\Gamma^2 d\sigma dt \\
&\quad - \int_{\Sigma_T} \theta^3 \xi^3 (\theta \xi)^{-3} \beta^2 z_\Gamma^2 d\sigma dt - 2\lambda R C \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - \gamma \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \\
&\quad - \lambda R C \int_{\Sigma_T} \theta \xi |\delta \Delta_\Gamma z_\Gamma|^2 d\sigma dt - \lambda R C \int_{\Sigma_T} \theta^3 \xi^3 (\theta \xi)^{-2} z_\Gamma^2 d\sigma dt.
\end{aligned} \tag{2.43}$$

Moreover, by definition of θ and ξ we also have that there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$,

$$|\theta \xi|^{-k} \leq C. \tag{2.44}$$

Thus using (2.44), we get from (2.43) that

$$\begin{aligned}
J &\geq \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{\lambda^2 R} - \frac{1}{\lambda^3 R^3} - \frac{1}{\lambda^2 R^2}\right) z_\Gamma^2 d\sigma dt - \lambda^2 R C \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt \\
&\quad - R \int_{\Sigma_T} \alpha_{tt} z_\Gamma^2 d\sigma dt - 2\lambda R C \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - \gamma \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \\
&\quad - \lambda R C \int_{\Sigma_T} \theta \xi |\delta \Delta_\Gamma z_\Gamma|^2 d\sigma dt.
\end{aligned} \tag{2.45}$$

Now, thanks to the boundary conditions in (2.6), we have

$$\begin{aligned}
-\gamma \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt &= -\frac{\gamma}{2} \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt - \frac{\gamma}{2} \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \\
&= -\frac{\gamma}{2} \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt - \frac{1}{2} \int_{\Sigma_T} |\partial_t z_\Gamma|^2 d\sigma dt - \frac{\delta}{2} \int_{\Sigma_T} \partial_t z_\Gamma \Delta_\Gamma z_\Gamma d\sigma dt \\
&\quad + \frac{1}{2} \int_{\Sigma_T} (\beta - R\alpha_t + R\gamma \partial_\nu \alpha) z_\Gamma \partial_t z_\Gamma d\sigma dt.
\end{aligned} \tag{2.46}$$

In addition, integrating the last term in the right hand side of (2.46) by parts in time yields

$$\begin{aligned}
\frac{1}{2} \int_{\Sigma_T} (\beta - R\alpha_t + R\gamma \partial_\nu \alpha) z_\Gamma \partial_t z_\Gamma d\sigma dt &= \frac{1}{4} \int_{\Sigma_T} (\beta - R\alpha_t + R\gamma \partial_\nu \alpha) \partial_t (z_\Gamma^2) d\sigma dt \\
&= -\frac{R}{4} \int_{\Sigma_T} (\gamma \partial_\nu \alpha_t - \alpha_{tt}) z_\Gamma^2 d\sigma dt.
\end{aligned}$$

Therefore, from (2.45) we obtain

$$\begin{aligned}
J \geq & \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{\lambda^2 R} - \frac{1}{\lambda^3 R^3} - \frac{1}{\lambda^2 R^2} \right) z_\Gamma^2 d\sigma dt - \lambda^2 RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt \\
& - \frac{3R}{4} \int_{\Sigma_T} \alpha_{tt} z_\Gamma^2 d\sigma dt - 2\lambda RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - \frac{\gamma}{2} \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt - \frac{R\gamma}{4} \int_{\Sigma_T} \partial_\nu \alpha_t z_\Gamma^2 d\sigma dt \\
& - \frac{1}{2} \int_{\Sigma_T} |\partial_t z_\Gamma|^2 d\sigma dt - \frac{\delta}{2} \int_{\Sigma_T} \partial_t z_\Gamma \Delta_\Gamma z_\Gamma d\sigma dt - \lambda RC \int_{\Sigma_T} \theta \xi |\delta \Delta_\Gamma z_\Gamma|^2 d\sigma dt. \tag{2.47}
\end{aligned}$$

Now, by definition of α we have $\partial_\nu \alpha_t = -\lambda \theta_t \xi \partial_\nu \eta$ and $|\partial_\nu \alpha_t| \leq \lambda C \theta^2 \xi^3$. Hence, using also (2.31) and (2.44), and for λ and R large enough, (2.47) becomes

$$\begin{aligned}
J \geq & \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{\lambda^2 R} - \frac{1}{\lambda^3 R^3} - \frac{1}{\lambda^2 R^2} - \frac{1}{\lambda^3 R^2} \right) z_\Gamma^2 d\sigma dt \\
& - \lambda^2 RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - 2\lambda RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - \lambda RC \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \\
& - \lambda RC \int_{\Sigma_T} \theta \xi \left(|\partial_t z_\Gamma|^2 + |\delta \Delta_\Gamma z_\Gamma|^2 + 2\partial_t z_\Gamma \Delta_\Gamma z_\Gamma \right) d\sigma dt \\
\geq & \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{\lambda^2 R} - \frac{1}{\lambda^3 R^3} - \frac{1}{\lambda^2 R^2} - \frac{1}{\lambda^3 R^2} \right) z_\Gamma^2 d\sigma dt - \lambda^2 RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt \\
& - 2\lambda RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt - 3\lambda RC \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt - 2\lambda RC \gamma \delta \int_{\Sigma_T} \Delta_\Gamma z_\Gamma \partial_\nu z d\sigma dt \\
& - \lambda RC \int_{\Sigma_T} \theta \xi \left(|\partial_t z_\Gamma|^2 + |\delta \Delta_\Gamma z_\Gamma|^2 + 2\partial_t z_\Gamma \Delta_\Gamma z_\Gamma - 2\partial_t z_\Gamma \partial_\nu z - 2\gamma \delta \Delta_\Gamma z_\Gamma \partial_\nu z \right) d\sigma dt. \tag{2.48}
\end{aligned}$$

Furthermore, using the Young inequality we obtain that

$$\begin{aligned}
& - \lambda^2 RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt \geq -\lambda^3 RC \int_{\Sigma_T} \theta \xi z_\Gamma^2 d\sigma dt - \lambda RC \int_{\Sigma_T} \theta \xi |\gamma \partial_\nu z|^2 d\sigma dt \\
& - 2\lambda RC \int_{\Sigma_T} \theta \xi z_\Gamma \partial_\nu z d\sigma dt \geq -\lambda RC \int_{\Sigma_T} \theta \xi z_\Gamma^2 d\sigma dt - \lambda RC \int_{\Sigma_T} \theta \xi |\gamma \partial_\nu z|^2 d\sigma dt \\
& - 3\lambda RC \int_{\Sigma_T} \partial_t z_\Gamma \partial_\nu z d\sigma dt \geq -C \int_{\Sigma_T} |\partial_t z_\Gamma|^2 d\sigma dt - C \int_{\Sigma_T} |\gamma \partial_\nu z|^2 d\sigma dt \\
& - 2\lambda RC \gamma \delta \int_{\Sigma_T} \Delta_\Gamma z_\Gamma \partial_\nu z d\sigma dt \geq -\lambda RC \int_{\Sigma_T} |\delta \Delta_\Gamma z_\Gamma|^2 d\sigma dt - \lambda RC \int_{\Sigma_T} |\delta \partial_\nu z|^2 d\sigma dt.
\end{aligned}$$

By means of these four expressions, from (2.48) we get that

$$\begin{aligned}
J \geq & \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{R^2} - \frac{2}{\lambda^2 R^2} - \frac{1}{\lambda^2 R} - \frac{1}{\lambda^3 R^2} - \frac{1}{\lambda^3 R^3} \right) z_\Gamma^2 d\sigma dt \\
& - \lambda RC \int_{\Sigma_T} \theta \xi \left(|\partial_t z_\Gamma|^2 + |\delta \Delta_\Gamma z_\Gamma|^2 + |\gamma \partial_\nu z|^2 + 2\delta \partial_t z_\Gamma \Delta_\Gamma z_\Gamma - 2\gamma \partial_t z_\Gamma \partial_\nu z - 2\gamma \delta \Delta_\Gamma z_\Gamma \partial_\nu z \right) d\sigma dt \\
= & \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 \left(1 - \frac{1}{R^2} - \frac{2}{\lambda^2 R^2} - \frac{1}{\lambda^2 R} - \frac{1}{\lambda^3 R^2} - \frac{1}{\lambda^3 R^3} \right) z_\Gamma^2 d\sigma dt \\
& - \lambda RC \int_{\Sigma_T} \theta \xi |\partial_t z_\Gamma + \delta \Delta_\Gamma z_\Gamma - \gamma \partial_\nu z|^2 d\sigma dt.
\end{aligned}$$

Hence, for λ and R large enough, we finally have

$$J \geq \lambda^3 R^3 C \int_{\Sigma_T} \theta^3 \xi^3 z_\Gamma^2 d\sigma dt - \lambda R C \int_{\Sigma_T} \theta \xi |\partial_t z_\Gamma + \delta \Delta_\Gamma z_\Gamma - \gamma \partial_\nu z|^2 d\sigma dt.$$

Now, collecting all the above estimates, we get from (2.9) that

$$\begin{aligned} & \lambda^3 R^2 C \int_{\Omega_T} \theta^3 \xi^3 z^2 dx dt + \lambda C \int_{\Omega_T} \theta \xi |\nabla z|^2 dx dt + \lambda^2 R^2 C \int_{\Sigma_T} \theta^3 \xi^3 z_\Gamma^2 d\sigma dt \\ & \leq C \int_{\Sigma_T} \theta \xi |\partial_t z_\Gamma + \delta \Delta_\Gamma z_\Gamma - \gamma \partial_\nu z|^2 d\sigma dt. \end{aligned} \quad (2.49)$$

Recall that $z = \phi e^{-R\alpha}$ so that using the fact that the functions α and η are constants at the boundary Γ , we get that

$$\begin{cases} \nabla z = & e^{-R\alpha} \left(\nabla \phi - R \phi \nabla \alpha \right) \\ \partial_\nu z = & e^{-R\alpha} \left(\partial_\nu \phi - R \phi \partial_\nu \alpha \right) \\ \partial_t z_\Gamma = & e^{-R\alpha} \left(\phi_t - R \phi \alpha_t \right) \\ \Delta_\Gamma z_\Gamma = & e^{-R\alpha} \Delta_\Gamma \phi_\Gamma. \end{cases} \quad (2.50)$$

Finally, coming back to the variable ϕ in (2.49) by using (2.50), we obtain the estimate (2.2) and the proof is finished. \square

We conclude this (sub)section with the following remark.

Remark 2.2. We notice that all the above estimates including the Carleman estimate (2.2) hold for $\delta = 0$.

2.2. The observability inequality. In this (sub)section, we give the last ingredient needed in the proof of our main result, namely we show an observability inequality.

Proposition 2.3. *Let $T > 0$ be fixed but arbitrary. Then there exists a constant $C_T > 0$ such that for every $(\phi_T, \phi_{\Gamma,T}) \in \mathbb{X}^2(\overline{\Omega})$, the unique mild solution ϕ of the backward system (1.11) satisfies the estimate*

$$\int_{\Omega} |\phi(x, 0)|^2 dx + \int_{\Gamma} |\phi_\Gamma(x, 0)|^2 d\sigma \leq C_T \int_{\Sigma_T} |\beta(x) \phi_\Gamma(x, t)|^2 d\sigma dt. \quad (2.51)$$

Proof. First, assume that $(\phi_T, \phi_{\Gamma,T}) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ and let ϕ be the unique strong solution of the backward system (1.11) with final data $(\phi_T, \phi_{\Gamma,T})$. Let $\lambda \geq \lambda_0$ and $R \geq R_0$ be fixed such that the estimate (2.2) holds. Then in particular, we have that

$$\begin{aligned} & \lambda^3 R^2 \int_{\Omega_T} \theta^3 \xi^3 e^{-2R\alpha} \phi^2 dx dt + \lambda^2 R^2 \int_{\Sigma_T} \theta^3 \xi^3 e^{-2R\alpha} \phi_\Gamma^2 d\sigma dt \\ & \leq C \int_{\Sigma_T} \theta \xi e^{-2R\alpha} |\partial_t \phi_\Gamma + \delta \Delta_\Gamma \phi_\Gamma - \gamma \partial_\nu \phi|^2 d\sigma dt. \end{aligned} \quad (2.52)$$

It is straightforward to check that there exist two positive constants \mathcal{P}_1 and \mathcal{P}_2 such that

$$\begin{cases} \theta^3 \xi^3 e^{-2R\alpha} \geq \mathcal{P}_1 & \text{in } \Omega \times [\frac{T}{4}, \frac{3T}{4}], \\ \theta \xi e^{-2R\alpha} \leq \mathcal{P}_2 & \text{on } \Sigma_T. \end{cases} \quad (2.53)$$

Using (2.53) we get from (2.52) that there is a constant $C > 0$ such that

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} \phi^2 dx dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Gamma} \phi_{\Gamma}^2 d\sigma dt \leq C \int_{\Sigma_T} |\partial_t \phi_{\Gamma} + \delta \Delta_{\Gamma} \phi_{\Gamma} - \gamma \partial_{\nu} \phi|^2 d\sigma dt.$$

Second, multiplying the first two equations in (1.11) by ϕ and ϕ_{Γ} , and integrating over Ω and Γ , respectively, we obtain that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \phi^2 dx = \gamma \int_{\Omega} |\nabla \phi|^2 dx - \gamma \int_{\Gamma} \phi_{\Gamma} \partial_{\nu} \phi d\sigma \quad (2.54)$$

and

$$\frac{1}{2} \frac{d}{dt} \int_{\Gamma} \phi_{\Gamma}^2 d\sigma = \delta \int_{\Gamma} |\nabla_{\Gamma} \phi_{\Gamma}|^2 d\sigma + \gamma \int_{\Gamma} \phi_{\Gamma} \partial_{\nu} \phi d\sigma + \int_{\Gamma} \beta \phi_{\Gamma}^2 d\sigma. \quad (2.55)$$

Adding (2.54) and (2.55) and using (1.7) we get that there is a constant $C > 0$ such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} \phi^2 dx + \int_{\Gamma} \phi_{\Gamma}^2 d\sigma \right) &= \gamma \int_{\Omega} |\nabla \phi|^2 dx + \delta \int_{\Gamma} |\nabla_{\Gamma} \phi_{\Gamma}|^2 d\sigma + \int_{\Gamma} \beta \phi_{\Gamma}^2 d\sigma \\ &\geq C \left(\int_{\Omega} \phi^2 dx + \int_{\Gamma} \phi_{\Gamma}^2 d\sigma \right). \end{aligned}$$

This clearly implies that

$$e^{CT} \left(\int_{\Omega} |\phi(x, 0)|^2 dx + \int_{\Gamma} |\phi_{\Gamma}(x, 0)|^2 d\sigma \right) \leq \int_{\Omega} \phi^2 dx + \int_{\Gamma} \phi_{\Gamma}^2 d\sigma. \quad (2.56)$$

Integrating (2.56) in time from $\frac{T}{4}$ to $\frac{3T}{4}$ we get that

$$\frac{T}{2} e^{CT} \left(\int_{\Omega} |\phi(x, 0)|^2 dx + \int_{\Gamma} |\phi_{\Gamma}(x, 0)|^2 d\sigma \right) \leq \int_{\frac{T}{4}}^{\frac{3T}{4}} \left(\int_{\Omega} \phi^2 dx + \int_{\Gamma} \phi_{\Gamma}^2 d\sigma \right) dt.$$

Thus, we obtain the observability inequality

$$\begin{aligned} \int_{\Omega} |\phi(x, 0)|^2 dx + \int_{\Gamma} |\phi_{\Gamma}(x, 0)|^2 d\sigma &\leq \frac{2C_1}{T} e^{-CT} \int_{\Sigma_T} |\partial_t \phi_{\Gamma} + \delta \Delta_{\Gamma} \phi_{\Gamma} - \gamma \partial_{\nu} \phi|^2 d\sigma dt \\ &= \frac{2C_1}{T} e^{-CT} \int_{\Sigma_T} |\beta \phi_{\Gamma}|^2 d\sigma dt \end{aligned}$$

for a strong solution.

Finally, let $(\phi_T, \phi_{\Gamma, T}) \in \mathbb{X}^2(\overline{\Omega})$ and ϕ the unique mild solution of the backward system (1.11). Let $(\phi_{T, n}, \phi_{\Gamma, T, n}) \in \mathbb{W}_{\delta}^{1,2}(\overline{\Omega})$ be a sequence which converges to $(\phi_T, \phi_{\Gamma, T})$ in $\mathbb{X}^2(\overline{\Omega})$. Then the strong solution $(\phi_n, \phi_{n, \Gamma})$ with final data $(\phi_{T, n}, \phi_{\Gamma, T, n})$ converges in $C([0, T]; \mathbb{X}^2(\overline{\Omega}))$ to the mild solution (ϕ, ϕ_{Γ}) with final data $(\phi_T, \phi_{\Gamma, T})$. It follows from the first part of the proof that

$$\int_{\Omega} \phi_n(x, 0)^2 dx + \int_{\Gamma} \phi_{n, \Gamma}^2(x, 0) d\sigma \leq \frac{2C_1}{T} e^{-CT} \int_{\Sigma_T} |\beta \phi_{n, \Gamma}|^2 d\sigma dt. \quad (2.57)$$

Taking the limit of (2.57) as $n \rightarrow \infty$ and using the above mentioned convergence, we get the estimate (2.51) and the proof is finished. \square

Now we are ready to give the proof of the main result of the paper.

Proof of Theorem 1.6. We use some ideas of the proof of [13, Theorem 4.2]. Let us introduce the following weighted L^2 -spaces

$$\begin{aligned} Z_{\Omega_T} &:= \left\{ f \in L^2(\Omega_T) : e^{R\alpha}(\theta\xi)^{-3/2} f \in L^2(\Omega_T) \right\}, \quad \langle f_1, f_2 \rangle_{Z_{\Omega_T}} = \int_{\Omega_T} f_1 f_2 e^{2R\alpha}(\theta\xi)^{-3} dx dt, \\ Z_{\Sigma_T} &:= \left\{ g \in L^2(\Sigma_T) : e^{R\alpha}(\theta\xi)^{-3/2} g \in L^2(\Sigma_T) \right\}, \quad \langle g_1, g_2 \rangle_{Z_{\Sigma_T}} = \int_{\Sigma_T} g_1 g_2 e^{2R\alpha}(\theta\xi)^{-3} d\sigma dt. \end{aligned}$$

The boundary controllability of the system (1.1) will follow by a duality argument. At this purpose, let us define the bounded linear operator $\mathcal{T} : L^2(\Sigma_T) \rightarrow \mathbb{X}^2(\overline{\Omega})$ by

$$\mathcal{T}v := \int_0^T e^{(T-\tau)A_\delta}(0, -v(\tau)) d\tau,$$

where we recall that $(e^{tA_\delta})_{t \geq 0}$ is the strongly continuous submarkovian semigroup generated by the operator A_δ in $\mathbb{X}^2(\overline{\Omega})$. Using the continuous embedding $Z_{\Omega_T} \times Z_{\Sigma_T} \hookrightarrow L^2(\Omega_T) \times L^2(\Sigma_T)$, we also introduce the bounded linear operator $\mathcal{S} : \mathbb{X}^2(\overline{\Omega}) \times Z_{\Omega_T} \times Z_{\Sigma_T} \rightarrow \mathbb{X}^2(\overline{\Omega})$ given by

$$\mathcal{S}(U_0, f, g) := e^{TA_\delta}U_0 + \int_0^T e^{(T-\tau)A_\delta}(f(\cdot, \tau), g(\cdot, \tau)) d\tau.$$

We claim that

$$\mathcal{S}(U_0, 0, g) - \mathcal{T}v = (u(\cdot, T), u_\Gamma(\cdot, T)), \quad (2.58)$$

where u is the unique mild solution of the system

$$\begin{cases} u_t - \gamma \Delta u = 0 & \text{in } \Omega \times (0, T) \\ \partial_t u_\Gamma - \delta \Delta_\Gamma u_\Gamma + \gamma \partial_\nu u + \beta u = g + v & \text{on } \Gamma \times (0, T) \\ (u, u_\Gamma)|_{t=0} = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \quad (2.59)$$

In fact, using Proposition 1.3 and the representation of mild solution given in Definition 1.2, we have that

$$\begin{aligned} \mathcal{S}(U_0, 0, g) - \mathcal{T}v &= e^{TA_\delta}U_0 + \int_0^T e^{(T-\tau)A_\delta}(0, g(\cdot, \tau)) d\tau - \int_0^T e^{(T-\tau)A_\delta}(0, -v(\cdot, \tau)) d\tau \\ &= e^{TA_\delta}U_0 + \int_0^T e^{(T-\tau)A_\delta}(0, g(\cdot, \tau) + v(\cdot, \tau)) d\tau = (u(\cdot, T), u_\Gamma(\cdot, T)), \end{aligned}$$

and we have shown the claim. Furthermore it is straightforward to check that the adjoint operator $\mathcal{T}^* : \mathbb{X}^2(\overline{\Omega}) \rightarrow L^2(\Sigma_T)$ is given for $\Phi_T := (\phi_T, \phi_{\Gamma, T})$ by

$$\mathcal{T}^*\Phi_T = -\phi_\Gamma$$

where $(\phi(t), \phi_\Gamma(t)) = e^{(T-t)A_\delta}(\phi_T, \phi_{\Gamma, T})$ is the solution of the backward system (1.11) with final data $(\phi_T, \phi_{\Gamma, T})$, while the adjoint operator $\mathcal{S}^* : \mathbb{X}^2(\overline{\Omega}) \rightarrow \mathbb{X}^2(\overline{\Omega}) \times Z_{\Omega_T} \times Z_{\Sigma_T}$ of \mathcal{S} is given by

$$\mathcal{S}^*\Phi_T = ((\phi(0), \phi_\Gamma(0)), e^{-2R\alpha}(\theta\xi)^3 \phi, e^{-2R\alpha}(\theta\xi)^3 \phi_\Gamma).$$

Now, the observability inequality (2.51) and the Carleman estimate (2.2) imply that

$$\begin{aligned} \|\mathcal{S}^* \phi_T\|_{\mathbb{X}^2(\overline{\Omega}) \times Z_{\Omega_T} \times Z_{\Sigma_T}}^2 &= \|(\phi(\cdot, 0), \phi_\Gamma(\cdot, 0))\|_{\mathbb{X}^2(\overline{\Omega})}^2 + \int_{\Omega_T} \theta^3 \xi^3 e^{-2R\alpha} \phi^2 dx dt \\ &\quad + \int_{\Sigma_T} \theta^3 \xi^3 e^{-2R\alpha} \phi_\Gamma^2 d\sigma dt \\ &\leq C_T \int_{\Sigma_T} \beta^2 |\phi_\Gamma|^2 d\sigma dt \leq C_T \|\beta\|_{L^\infty(\Gamma)}^2 \|\mathcal{T}^* \Phi_T\|_{L^2(\Sigma_T)}, \end{aligned} \quad (2.60)$$

at first for $(\phi_T, \phi_{\Gamma,T}) \in \mathbb{W}_\delta^{1,2}(\overline{\Omega})$ and then for $(\phi_T, \phi_{\Gamma,T}) \in \mathbb{X}^2(\overline{\Omega})$ by using an approximation argument as at the end of the proof of Proposition 2.3. By [18, Theorem IV.2.2], the estimate (2.60) implies that $\text{Im}(\mathcal{S}) \subset \text{Im}(\mathcal{T})$. This shows that for every $(U_0, f, g) \in \mathbb{X}^2(\overline{\Omega}) \times Z_{\Omega_T} \times Z_{\Sigma_T}$, there exists a control $v \in L^2(\Sigma_T)$ such that $\mathcal{S}(U_0, f, g) = \mathcal{T}v$. Therefore,

$$(u(\cdot, T), u_\Gamma(\cdot, T)) = \mathcal{S}(U_0, f, g) - \mathcal{T}v = (0, 0)$$

and the proof is finished. \square

REFERENCES

- [1] W. Arendt, G. Metafun, D. Pallara and S. Romanelli, *The Laplacian with Wentzell-Robin boundary conditions on spaces of continuous functions*. Semigroup Forum **67** (2003), 247–261.
- [2] H. Brezis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*. North Holland Mathematics Studies, vol. 5, North-Holland, Amsterdam, London, 1973.
- [3] E.B. Davies, *Heat kernels and Spectral Theory*. Cambridge University Press, Cambridge, 1990.
- [4] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, *The heat equation with generalized Wentzell boundary condition*. J. Evol. Equ. **2** (2002), 1–19.
- [5] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, *The heat equation with nonlinear general Wentzell boundary condition*. Adv. Differential Equations **11** (2006), 481–510.
- [6] E. Fernández-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*. SIAM J. Control Optim. **45** (2006), 1399–1446.
- [7] C.G. Gal and M. Warma, *Well posedness and the global attractor of some quasi-linear parabolic equations with nonlinear dynamic boundary conditions*. Differential Integral Equations **23** (2010), 327–358.
- [8] G. Goldstein, *Derivation and physical interpretation of general boundary conditions*. Adv. Differential Equations **11** (2006), 457–480.
- [9] D. Hömberg, K. Krumbiegel and J. Rehberg, *Optimal control of a parabolic equation with dynamic boundary condition*. Appl. Math. Optim. **67** (2013), 3–31.
- [10] O. Yu. Imanuilov, *Boundary controllability of parabolic equations*. Uspekhi Mat. Nauk **48** (1993), 211–212; translation in Russian Math. Surveys **48** (1993), 192–194.
- [11] J. Jost, *Riemannian geometry and geometric analysis*. Fifth edition. Springer-Verlag, Berlin, 2008.
- [12] M. Kumpf and G. Nickel, *Dynamic boundary conditions and boundary control for the one-dimensional heat equation*. J. Dynam. Control Systems **10** (2004), 213–225.
- [13] L. Maniar, M. Meyries and R. Schnaubelt, *Null controllability for parabolic equations with dynamic boundary conditions of reactive-diffusion type*. arXiv:1311.076, 2013.
- [14] M.E. Taylor, *Partial Differential Equations. Basic theory*. Springer-Verlag, New York, 1996.
- [15] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*. J.A. Barth, Heidelberg, 1995.
- [16] J.-J. Vázquez and E. Vitillaro, *Heat equation with dynamical boundary conditions of reactive-diffusive type*. J. Differential Equations **250** (2011), 2143–2161.
- [17] M. Warma, *Parabolic and elliptic problems with general Wentzell boundary condition on Lipschitz domains*. Commun. Pure Appl. Anal. **12** (2013), 1881–1905.
- [18] J. Zabczyk, *Mathematical Control Theory. An Introduction*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008.

U. BICCARI, BASQUE CENTER FOR APPLIED MATHEMATICS (BCAM), ALAMEDA MAZARREDO 14. 48009 BILBAO
BASQUE COUNTRY (SPAIN)

E-mail address: `ubiccari@bcamath.org`

M. WARMA, UNIVERSITY OF PUERTO RICO, FACULTY OF NATURAL SCIENCES, DEPARTMENT OF MATHEMATICS
(RIO PIEDRAS CAMPUS), PO BOX 70377 SAN JUAN PR 00936-8377 (USA)

E-mail address: `mahamadi.warma1@upr.edu`, `mjwarma@gmail.com`